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Derivation of the total twist from Chern-Simons theory

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Abstract

The total twist number, which represents the first non-trivial Vassiliev knot invariant, is derived from the second order expression of the Wilson loop expectation value in the Chern-Simons theory. Using the well-known fact that the analytical expression is an invariant, a non-recursive formulation of the total twist based on the evaluation of knot diagrams is constructed by an appropriate deformation of the knot line in the three-dimensional Euclidian space. The relation to the original definition of the total twist is elucidated.

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1 Introduction

The use of Chern-Simons theory as a tool for knot theory has proved to be very fruitful. Some of the background of the relationship of these two areas is recalled very briefly in order to fix the starting point for the present work.

One considers the Chern-Simons quantum field theory on the manifold S^3 with some gauge group, say $SU(N)$. It is characterized by a generating functional

$$Z[J] = \int \mathcal{D}A \ \exp\left\{ iS_{\mathrm{CS}}[A] + \int_{S^3} d^3x \ J_a^\mu(x) A_\mu^a(x) \right\}, \tag{1.1}$$

where S_{CS} is the integral of the Chern-Simons differential form:

$$S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{S^3} \mathrm{tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\} \tag{1.2}$$

and $A = A_\mu^a T_a dx^\mu$ is a gauge field in some appropriately normalized representation T. Here k must be an integer in order to maintain gauge invariance. The observables of the theory are Wilson-line operators along framed links. The concept of framing was introduced in the original work of Witten [1]. The Wilson loops are symbolically written as path-ordered exponentials of line integrals along knots:

$$W(K) = \mathrm{tr} P \exp\left\{ \oint_K A \right\} \qquad \text{for a knot } K \text{ embedded in } S^3, \tag{1.3}$$

$$W(L) = W(K_1) \ldots W(K_n) \qquad \text{for a link with } n \text{ components } L = \{K_1, \ldots, K_n\}. \tag{1.4}$$

In the framework of quantum field theory the expectation values of $W(L)$ are defined as

$$\langle W(L) \rangle_f = \frac{\int \mathcal{D}A \ W(L) \exp\{iS_{\mathrm{CS}}[A]\}}{\int \mathcal{D}A \ \exp\{iS_{\mathrm{CS}}[A]\}}, \tag{1.5}$$

where f symbolizes a framing prescription used in the evaluation of the Wilson loop. The expectation values are dependent on the coupling constant k and on the order N of the gauge group (and on the framing f). It has been shown that $\langle W(L) \rangle$ associated to a link L fulfils the skein relation of a so-called generalized Jones polynomial. A comprehensive exposition of this subject can be found in a monograph by E. Guadagnini [2].

The Wilson loops can be expanded in a perturbation series. Then $\langle W(L) \rangle$ is a power series in $1/k$ and the Casimir factors of the gauge group $c_2(T)$, c_v, $c_4(T)$ etc. Its coefficients are multi-dimensional path-ordered line integrals along the knot. Since $1/k$ and the Casimir factors can be considered as linearly independent the coefficient of every monomial is a link invariant. These line integrals can be calculated either numerically or by expanding the generalized Jones polynomial mentioned above, as demonstrated in a recent paper by Alvarez and Labastida [3].

We present here a direct method which does not rely on recursive procedures. Our starting point is the analytical expression for $\langle W(K) \rangle$ in the second order of $1/k$ which has been calculated by Guadagnini et al. in [4]:

$$\langle W(K) \rangle_{f,2.\mathrm{order}} = \dim T \left(\frac{2\pi}{k} \right)^2 \left\{ -\frac{1}{2} c_2^2(T) \varphi_f^2(K) + c_v c_2(T) [\rho_1(K) + \rho_2(K)] \right\}, \tag{1.6}$$

where $\varphi_f(K)$ is the framing number of the framed knot. The line integrals $\rho_1(K)$ and $\rho_2(K)$ are given by

$$\begin{aligned}\rho_1(K) &= -\frac{1}{32\pi^3}\int_K dx_1^{\mu_1}\int_{\mathrm{BP}}^{x_1} dx_2^{\mu_2}\int_{\mathrm{BP}}^{x_2} dx_3^{\mu_3}\ \epsilon^{\nu_1\nu_2\nu_3}\epsilon_{\mu_1\nu_1\sigma_1}\epsilon_{\mu_2\nu_2\sigma_2}\epsilon_{\mu_3\nu_3\sigma_3}\\ &\times \int d^3z\frac{(z-x_1)^{\sigma_1}}{|z-x_1|^3}\frac{(z-x_2)^{\sigma_2}}{|z-x_2|^3}\frac{(z-x_3)^{\sigma_3}}{|z-x_3|^3}\end{aligned} \tag{1.7}$$

$$\begin{aligned}\rho_2(K) &= \frac{1}{8\pi^2}\int_K dx_1^{\mu_1}\int_{\mathrm{BP}}^{x_1} dx_2^{\mu_2}\int_{\mathrm{BP}}^{x_2} dx_3^{\mu_3}\int_{\mathrm{BP}}^{x_3} dx_4^{\mu_4}\ \ \epsilon_{\mu_4\mu_2\sigma_2}\epsilon_{\mu_3\mu_1\sigma_1}\\ &\times \frac{(x_4-x_2)^{\sigma_2}}{|x_4-x_2|^3}\frac{(x_3-x_1)^{\sigma_1}}{|x_3-x_1|^3}.\end{aligned} \tag{1.8}$$

They are frame-independent, as shown in [4]. The knot invariant we are interested in is

$$\rho^{\mathrm{II}}(K) := \rho_1(K) + \rho_2(K). \tag{1.9}$$

Our method to calculate ρ^{II} is based on the evaluation of knot diagrams $\mathcal{K}$ only, without recurse to skein relations. The program can be summed up in the following steps.

- We are interested of course in the evaluation of ρ^{II} for arbitrary knots embedded in three-dimensional space. To achieve this it is convenient to work with the knot diagrams. The formalism appropriate for our purposes is developed in sections 2 to 4.

- The sense in which knot diagrams are flattened knots is clarified in section 5.

- The integral ρ_2 is calculated for the limit of flat knots. The result is an expression for ρ_2 based on the evaluation of knot diagrams (section 6).

- A similar expression for ρ_1 is constructed using some properties of the line integral $\rho_1(K)$ *and the invariance of* $\rho^{\mathrm{II}}(K)$. The knot theoretical formulation of $\rho^{\mathrm{II}}(\mathcal{K})$ is thereby completed (section 7).

- It is known from [4] that $\rho^{\mathrm{II}}(K)$ is closely related to the total twist τ defined by Lickorish and Millet in [5]. This relation is examined for our diagrammatical version of ρ^{II} (section 8).

- The procedure for calculating the invariants ρ^{II} and τ for a given knot using our expressions (7.14) and (8.1) , which are based on the crossing numbers introduced in section 4.1, is illustrated in section 9.

We should emphasize that the important point in this paper is not just to find another expression for the total twist, but to find a method of constructing it which can be generalized to other link invariants. As has been argued in [3] the line integrals from the Wilson loop expansion are Vassiliev invariants, if normalized correctly. These have been introduced by Vassiliev in [6]. For an axiomatic approach to this topic see Birman and Lin [7]. The total twist, refered to above, is the simplest non-trivial invariant of this kind [8]. The method presented here, extended to higher orders, should allow interesting insights concerning this large class of knot invariants.

2 Definition of the knot diagrams

In this section we shall give the definitions of the knot diagrams used in this paper. Based, oriented knots are considered. Their diagrams will be described as mappings from a circle to a plane. First, define the unit interval $I = [0, 1]$ and an equivalence relation which identifies $0 \sim 1$. Then $I_{/\sim}$ is homeomorphic to a circle. The projection plane is $\mathbb{R}^2$. The shadow diagram of some knot is given by a mapping

$$\pi : I_{/\sim} \to \mathbb{R}^2. \tag{2.1}$$

The point $\pi(0) = \pi(1)$ is called the basepoint. The diagram itself is

$$K = \mathrm{image}(\pi) \subset \mathbb{R}^2. \tag{2.2}$$

The set of crossings is given by

$$\mathcal{C} = \{x \in \mathbb{R}^2 \mid \exists\, t_1 \neq t_2 \in I_{/\sim} : \pi(t_1) = \pi(t_2) = x\} \subset \mathbb{R}^2. \tag{2.3}$$

The over/under crossing information is given by a function

$$\epsilon : \mathcal{C} \to \{-1, +1\}, \tag{2.4}$$

defined as in the accompanying figure. Then $\pi^{-1}(\mathcal{C})$ is a set $\{s_0, \ldots, s_{2n-1}\}$, with $s_i \in I_{/\sim}$ and n the number of crossings. More generally, we set $\pi^{-1}(\mathcal{C}) = \{s_i \mid i \in \mathcal{I}\}$, where $\mathcal{I}$ is some index set. The mapping π may be chosen such that for all $i \in \mathcal{I}$ $s_i \not\sim 0$, i.e. the basepoint does not coincide with any crossing. The knot diagram is now defined by

$$\mathcal{K} = (\pi, \epsilon, \mathcal{I}). \tag{2.5}$$

The order relations $>$ and $<$ in I induce relations $>$ and $<$ in the index set $\mathcal{I}$ according to

$$s_i > s_j \Rightarrow i > j, \quad i, j \in \mathcal{I}. \tag{2.6}$$

If not denoted otherwise $\mathcal{I} = \{0, \ldots, 2n - 1\}$ with its common order will be used. In this case the addition (modulo $2n$) of elements of $\mathcal{I}$ with integers is defined and will be used.

On $I_{/\sim}$ intervals will be denoted as follows. For $s_i \not\sim s_j$, $[s_i, s_j]$ is the closed arc from s_i to s_j following the orientation of $I_{/\sim}$. Hence

$$[s_i, s_j] \cup [s_j, s_i] = I_{/\sim} \text{ and } [s_i, s_j] \cap [s_j, s_i] = \{s_i, s_j\}. \tag{2.7}$$

From this, *ordered* subsets of the index set are defined, and denoted as

$$[i_a, i_b] = \{i \in \mathcal{I} \mid s_i \in [s_{i_a}, s_{i_b}]\} \text{ for } i_a \neq i_b, \tag{2.8}$$

with the property

$$[i_a, i_b] \cup [i_b, i_a] = \mathcal{I} \text{ and } [i_a, i_b] \cap [i_b, i_a] = \{i_a, i_b\}. \tag{2.9}$$

We define *open* subsets of the index set by

$$]i_a, i_b[= [i_a, i_b] \setminus \{i_a, i_b\}. \tag{2.10}$$

The following alternative notations for the crossing function will also be used:

$$\begin{array}{lll} \text{For all } i_1 \in \mathcal{I} & \text{define} & \epsilon(i_1) := \epsilon(\pi(s_{i_1})) \\ \text{For all } i_1, i_2 \in \mathcal{I} & \text{define} & \epsilon(i_1, i_2) := \begin{cases} \epsilon(i_1) \text{ if } \pi(s_{i_1}) = \pi(s_{i_2}) \text{ and } i_1 \neq i_2 \\ 0 \text{ otherwise} \end{cases} \end{array} \tag{2.11}$$

An example for a knot diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ defined in this way is shown in figure **??**. Here, $n = 4$ (figure-eight knot) and $\mathcal{I} = \{0, 1, \ldots, 7\}$.

Definition of pieces of the knot diagram. We consider a knot diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$. A piece of the knot diagram will be defined as an open connection between two crossings or one crossing with itself. Consider two indices $i_1 \neq i_2 \in \mathcal{I}$. Then a set such as

$$S = \pi(\ ]s_{i_1}, s_{i_2}[\ ) \subset \mathbb{R}^2 \tag{2.12}$$

is called a piece of the knot diagram. To this piece an index subset is associated:

$$\mathcal{I}(S) =]i_1, i_2[= \{i_1 + 1, i_1 + 2, \ldots, i_2 - 1\}. \tag{2.13}$$

Of course, a piece is not necessarily free of self-crossings.

Two pieces S and T will be called non-overlapping if $\mathcal{I}(S) \cap \mathcal{I}(T) = \emptyset$. In this case S and T can intersect each other, but the set of common points only consists of crossings.

3 Description of the Reidemeister moves

It is necessary for the following to formulate the Reidemeister moves in terms of π, ϵ, and $\mathcal{I}$. In this section $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ will always denote the the knot diagram before the Reidemeister move, and $\mathcal{K}' = (\pi', \epsilon', \mathcal{I}')$ will be the diagram afterwards. The sets of

crossings are denoted by $\mathcal{C}$ and $\mathcal{C}'$. The Reidemeister moves can be formulated in the following way. In the diagram $L_{\rm I}^{+/-}$, the indices which form the crossing are k_1 and k_1+1, and the situation can obviously be described by

$$\pi(s_{k_1}) = \pi(s_{k_1+1}). \tag{3.1}$$

After performing the Reidemeister-I move (see diagram $L_{\rm I}^0$ in figure (**??**)), i.e. replacing π by a mapping π', the situation is described by a new index set $\mathcal{I}'$ and a new crossing function $\epsilon' : \mathcal{C}' \to \{-1, +1\}$, where $\mathcal{C}' = \mathcal{C} \setminus \{\pi(s_k)\}$.

$$\mathcal{I}' = \mathcal{I} \setminus \{k_1, k_1+1\}, \;\; \epsilon' = \epsilon \mid_{\mathcal{C}'} . \tag{3.2}$$

For the second Reidemeister move two cases have to be distinguished. First we consider the situation $L_{\rm II-A}$ in figure **??** with the indices k_1, k_1+1, k_2, and k_2+1. It is characterized by

$$\begin{aligned} \pi(s_{k_1}) = \pi(s_{k_2}), \;\; \pi(s_{k_1+1}) = \pi(s_{k_2+1}), \\ \epsilon(k_1, k_2) = -\epsilon(k_1+1, k_2+1). \end{aligned} \tag{3.3}$$

In the situation $L_{\rm II-A}^0$, i.e. after the Reidemeister-II-A move, we have

$$\mathcal{I}' = \mathcal{I} \setminus \{k_1, k_1+1, k_2, k_2+1\}, \;\;\; \epsilon' = \epsilon \mid_{\mathcal{C}'} . \tag{3.4}$$

For $L_{\rm II-B}$ (see figure **??**) with the same indices as in $L_{\rm II-A}$ we have

$$\begin{aligned} \pi(s_{k_1}) = \pi(s_{k_2+1}), \;\; \pi(s_{k_2}) = \pi(s_{k_1+1}), \\ \epsilon(k_1, k_2+1) = -\epsilon(k_1+1, k_2). \end{aligned} \tag{3.5}$$

Performing the Reidemeister-II-B move we get the same $\mathcal{I}'$ and ϵ' as for the Reidemeister-II-A move.

A Reidemeister-III situation as shown in figure (**??**) with indices k_1, k_1+1, k_2, k_2+1, k_3, and k_3+1, and a cyclic orientation is described by

$$\begin{aligned} \pi(s_{k_1}) = \pi(s_{k_3+1}), \;\;\; \pi(s_{k_2}) = \pi(s_{k_1+1}), \;\;\; \pi(s_{k_3}) = \pi(s_{k_2+1}) \\ \epsilon(k_1)\epsilon(k_2) + \epsilon(k_2)\epsilon(k_3) + \epsilon(k_3)\epsilon(k_1) = -1. \end{aligned} \tag{3.6}$$

The Reidemeister-III move does not change the index set, i.e. $\mathcal{I}' = \mathcal{I}$. The relation between the crossing functions ϵ and ϵ' is:

$$\epsilon'(k_1) = \epsilon(k_2), \;\;\; \epsilon'(k_2) = \epsilon(k_3), \;\;\; \epsilon'(k_3) = \epsilon(k_1). \tag{3.7}$$

The relation (3.6) is also valid for ϵ'.

4 Crossing numbers

4.1 Definitions

We shall now define some functions of knot diagrams. Later they will be used to formulate the invariant ρ^{II} for diagrams. Let $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ be a knot diagram, with the set of crossings $\mathcal{C}$ and the number of crossings n. The behaviour of n under the Reidemeister moves can be easily found by counting the crossings. It is

$$\begin{aligned} n(L_{\mathrm{I}}^{+/-}) - n(L_{\mathrm{I}}^{0}) &= 1 \\ n(L_{\mathrm{II-A/B}}) - n(L_{\mathrm{II-A/B}}^{0}) &= 2 \\ n(L_{\mathrm{III}}^{+}) - n(L_{\mathrm{III}}^{-}) &= 0. \end{aligned} \tag{4.1}$$

The writhe number, which is a regular isotopy invariant, is defined by

$$\chi_1(\mathcal{K}) = \sum_{c \in \mathcal{C}} \epsilon(c). \tag{4.2}$$

This can also be written as

$$\chi_1(\mathcal{K}) = \sum_{\substack{j_1, j_2 \in \mathcal{I} \\ j_1 > j_2}} \epsilon(j_1, j_2). \tag{4.3}$$

For two non-overlapping pieces S and T of the diagram the first crossing number is defined as

$$\chi_1(S, T) = \sum_{i_S \in \mathcal{I}(S)} \sum_{i_T \in \mathcal{I}(T)} \epsilon(i_S, i_T). \tag{4.4}$$

Instead of writing $\chi_1(S, T)$ the notation using index sets $\chi_1(\mathcal{I}(S), \mathcal{I}(T)) := \chi_1(S, T)$ will also be used. If S and T start and end at the same point c (see figure **??**), $\chi_1(S, T)$ is closely related to the linking number λ of the two link components arising from *nullifying* c. Denoting these components as L_S and L_T one has

$$\frac{1}{2}\chi_1(S, T) = \lambda(L_S, L_S). \tag{4.5}$$

Finally, we define an object called the *second self-crossing number* which is *not* an invariant:

$$\chi_2(\mathcal{K}) = \sum_{\substack{j_1, j_2, j_3, j_4 \in \mathcal{I} \\ j_1 > j_2 > j_3 > j_4}} \epsilon(j_1, j_3)\epsilon(j_2, j_4). \tag{4.6}$$

It can easily be shown that χ_2 can also be written as

$$\chi_2(\mathcal{K}) = \frac{1}{4} \sum_{j_1, j_3 \in \mathcal{I}} \epsilon(j_1, j_3)\chi_1(\ ]j_1, j_3[\ ,\ ]j_3, j_1[\ ). \tag{4.7}$$

In this last formulation neither the basepoint nor the order of the elements of $\mathcal{I}$, which was fixed by the basepoint, appear. Therefore $\chi_2(\mathcal{K})$ is independent of the basepoint. Now some properties of $\chi_2(\mathcal{K})$ will be examined.

4.2 Behaviour of χ_2 under change of one crossing

An important property of χ_2, which will be used later, is its behaviour if the sign of one crossing is changed. Consider two knot diagrams $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ and $\mathcal{K}' = (\pi, \epsilon', \mathcal{I})$ which differ only in one crossing c, i.e.

$$\epsilon'(d) = \begin{cases} -\epsilon(d) & \text{if } d = c \\ +\epsilon(d) & \text{if } d \neq c. \end{cases} \tag{4.8}$$

The crossing is formed by two indices $i_c^+, i_c^- \in \mathcal{I}$. We want to calculate $\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}')$. It is convenient to use the representation (4.7). The range of summation $\mathcal{I}$ is divided into $\{i_c^+, i_c^-\}$ and $\mathcal{I} \setminus \{i_c^+, i_c^-\}$. Then

$$\begin{aligned} \chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') &= \frac{1}{2}\epsilon(i_c^+, i_c^-)\chi_1(\,]i_c^+, i_c^-[\,,\ ]i_c^-, i_c^+[\,;\epsilon) \\ &+ \frac{1}{4}\sum_{j_1, j_3 \in \mathcal{I}\setminus\{i_c^+, i_c^-\}} \epsilon(j_1, j_3)\chi_1(\,]j_1, j_3[\,,\ ]j_3, j_1[\,;\epsilon) \\ &- \frac{1}{2}\epsilon'(i_c^+, i_c^-)\chi_1(\,]i_c^+, i_c^-[\,,\ ]i_c^-, i_c^+[\,;\epsilon') \\ &- \frac{1}{4}\sum_{j_1, j_3 \in \mathcal{I}\setminus\{i_c^+, i_c^-\}} \epsilon'(j_1, j_3)\chi_1(\,]j_1, j_3[\,,\ ]j_3, j_1[\,;\epsilon'), \end{aligned} \tag{4.9}$$

where the additional argument of χ_1 indicates which crossing function is used. The first and the third summands give

$$\epsilon(i_c^+, i_c^-)\chi_1(\,]i_c^+, i_c^-[\,,\ ]i_c^-, i_c^+[\,;\epsilon). \tag{4.10}$$

In the second and fourth terms the ϵ-part can be factorized, since $\epsilon(j_1, j_3) = \epsilon'(j_1, j_3)$ in this subset of $\mathcal{I}$. The remaining factor is the difference of the χ_1-parts:

$$\begin{aligned} &\chi_1(\,]j_1, j_3[\,,\ ]j_3, j_1[\,;\epsilon) - \chi_1(\,]j_1, j_3[\,,\ ]j_3, j_1[\,;\epsilon') \\ &= \begin{cases} 2\epsilon(i_c^+, i_c^-) \text{ if } i_c^+ \in\,]j_1, j_3[\text{ and } i_c^- \in\,]j_3, j_1[\\ \qquad\qquad\ \ \text{or } i_c^- \in\,]j_1, j_3[\text{ and } i_c^+ \in\,]j_3, j_1[\\ 0 \text{ otherwise}. \end{cases} \end{aligned} \tag{4.11}$$

This follows immediately from the definition of χ_1 and the relation (4.8). The condition

$$i_c^+ \in\,]j_1, j_3[\quad \text{and} \quad i_c^- \in\,]j_3, j_1[\tag{4.12}$$

is equivalent to the condition

$$j_1 \in\,]i_c^-, i_c^+[\quad \text{and} \quad j_3 \in\,]i_c^+, i_c^-[\,. \tag{4.13}$$

Therefore the second and the fourth terms give

$$
\begin{aligned}
[\sum_{\substack{j_1 \in\,]i_c^+, i_c^-[\\ j_3 \in\,]i_c^-, i_c^+[}} + \sum_{\substack{j_1 \in\,]i_c^-, i_c^+[\\ j_3 \in\,]i_c^+, i_c^-[}}] \frac{1}{4}\epsilon(j_1, j_3) 2\epsilon(i_c^+, i_c^-) \\
= \chi_1(\ ]i_c^+, i_c^-[\ ,\ ]i_c^-, i_c^+[\ ; \epsilon)\epsilon(i_c^+, i_c^-).
\end{aligned}
\tag{4.14}
$$

Combining (4.10) und (4.14) we get

$$
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = 2\epsilon(i_c^+, i_c^-)\chi_1(\ ]i_c^+, i_c^-[\ ,\ ]i_c^-, i_c^+[\ ; \epsilon). \tag{4.15}
$$

4.3 Behaviour of χ_2 under the Reidemeister moves

In the following the behaviour of χ_2 under the moves RM-I, RM-II-A, RM-II-B, and RM-III will be examined. It is not necessary to calculate the behaviour for other versions of RM-III, e.g. with reversed orientations for some lines, because these moves can be composed from the moves mentioned above.

Behaviour of χ_2 under RM-I. Consider two knot diagrams $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ and $\mathcal{K}' = (\pi', \epsilon', \mathcal{I}')$ which are equal up to one Reidemeister-I move, so that $\mathcal{K}$ contains a situation $L_{\mathrm{I}}^{+/-}$ at the indices $k, k+1 \in \mathcal{I}$ and $\mathcal{K}'$ contains instead a situation L_{I}^0, as shown in figure **??**. This means that $\mathcal{I}' = \mathcal{I} \setminus \{k, k+1\}$. The basepoint is assumed to be outside the Reidemeister situation, i.e. somewhere in $\pi(\ ]s_{k+1}, s_k[\ )$, so that $k+1 \bmod 2n > k$. We now calculate $\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}')$. There are two sums over four summation variables, each of which satisfies $j_1 > j_2 > j_3 > j_4$. Since the sole difference between $\mathcal{K}$ and $\mathcal{K}'$ is the crossing at k and $k+1$, all cases in which none of the summation variables is k or $k+1$ cancel out. Consider the summands with $j_1 = k$. Due to the factor $\epsilon(j_1, j_3)$ the summands can only be non-vanishing if $j_3 = k+1$ which is not possible because the summation variables fulfil the condition $j_1 > j_3$. Consider now the summands with $j_1 = k+1$. Due to the factor $\epsilon(j_1, j_3)$ there can be a contribution only if $j_3 = k$. But in this case the condition $j_1 > j_2 > j_3$ cannot be fulfilled and hence there are no non-vanishing summands. The same argument is valid for all other cases. Therefore, the behaviour of χ_2 under the first Reidemeister move is

$$
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = 0. \tag{4.16}
$$

Behaviour of χ_2 under RM-II-A and RM-II-B. Consider a knot $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ which contains a situation $L_{\mathrm{II-A}}$ with the indices k_1, k_1+1, k_2, and k_2+1, where $k_2+1 > k_1$. Consider another knot $\mathcal{K}' = (\pi', \epsilon', \mathcal{I}')$ which is equal to $\mathcal{K}$ except for containing $L_{\mathrm{II-A}}^0$ instead of $L_{\mathrm{II-A}}$, as shown in figure **??**. We wish to calculate $\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}')$. Among all summands it is sufficient to consider those with at least one summation variable in $\{k_1, k_1+1, k_2, k_2+1\}$. However, if in the product $\epsilon(j_1, j_3)\epsilon(j_2, j_4)$ one of the arguments, e.g. j_1 belongs to $\{k_1, k_1+1, k_2, k_2+1\}$, the product is non-zero if and only if j_3 assumes

the corresponding value so that j_1 and j_3 form a crossing. This reduces the number of summands which have to be considered.

$$\begin{aligned}\chi_2(\mathcal{K})-\chi_2(\mathcal{K}') &= \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon(k_2,k_1)\epsilon(j_2,j_4)\\ &+ \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon(k_2+1,k_1+1)\epsilon(j_2,j_4)\\ &+ \sum_{\substack{j_1,j_3\in\mathcal{I}\\ j_1>k_2+1,k_2>j_3>k_1+1}} \epsilon(j_1,j_3)\epsilon(k_2,k_1)\\ &+ \sum_{\substack{j_1,j_3\in\mathcal{I}\\ j_1>k_2+1,k_2>j_3>k_1+1}} \epsilon(j_1,j_3)\epsilon(k_2+1,k_1+1)\\ &+ \epsilon(k_2+1,k_1+1)\epsilon(k_2,k_1)\end{aligned} \tag{4.17}$$

Using the relation (3.5) the first four terms cancel pairwise, and we are left with

$$\chi_2(\mathcal{K})-\chi_2(\mathcal{K}')=\epsilon(k_2+1,k_1+1)\epsilon(k_2,k_1)=-1. \tag{4.18}$$

For the Reidemeister-II-B move the calculation is similar, except that the only non-vanishing term is missing. Hence in this case

$$\chi_2(\mathcal{K})-\chi_2(\mathcal{K}')=0. \tag{4.19}$$

Behaviour of χ_2 under RM-III. The calculation of the behaviour of χ_2 under the third Reidemeister move is longer, but in principle no more complicated than for the preceding cases. As in the previous cases we consider two knots: $\mathcal{K}=(\pi,\epsilon,\mathcal{I})$ with a situation L^+_{III} consisting of the indices $\{k_1,k_1+1,k_2,k_2+1,k_3,k_3+1\}$ and $\mathcal{K}'=(\pi',\epsilon',\mathcal{I}')$ with a situation L^-_{III} at the same location, as shown in figure **??**. The case $k_3>k_2>k_1$ is considered. For simplicity we shall now assume[2] that the basepoint lies between the index k_3+1 and the next index $k_3+2 \bmod 2n$. In this situation there are no summands with variables $j_i>k_3+1$ because k_3+1 is the greatest index in $\mathcal{I}$. Only summands with two or four summation variables in $\{k_1,k_1+1,k_2,k_2+1,k_3,k_3+1\}$ can contribute.

$$\begin{aligned}\chi_2(\mathcal{K}) &- \chi_2(\mathcal{K}')\\ &= \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon(k_2,k_1+1)\epsilon(j_2,j_4)+\sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_1>j_4}} \epsilon(k_3,k_2+1)\epsilon(j_2,j_4)\\ &+ \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_2>j_4>k_1+1}} \epsilon(k_3,k_2+1)\epsilon(j_2,j_4)+\sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon(k_3+1,k_1)\epsilon(j_2,j_4)\end{aligned}$$

[2]This could have been done for the Reidemeister-II move as well. We renounced this in order to illustrate which terms can appear in sums of this type.

$$
\begin{aligned}
&+ \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_1>j_4}} \epsilon(k_3+1,k_1)\epsilon(j_2,j_4) + \sum_{\substack{j_1,j_3\in\mathcal{I}\\ k_3>j_1>k_2+1,k_2>j_3>k_1+1}} \epsilon(j_1,j_3)\epsilon(k_2,k_1+1)\\
&- \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon'(k_2+1,k_1)\epsilon'(j_2,j_4) - \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_1>j_4}} \epsilon'(k_3,k_1+1)\epsilon'(j_2,j_4)\\
&- \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_2>j_4>k_1+1}} \epsilon'(k_3+1,k_2)\epsilon'(j_2,j_4) - \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_2>j_2>k_1+1,k_1>j_4}} \epsilon'(k_3,k_1+1)\epsilon'(j_2,j_4)\\
&- \sum_{\substack{j_2,j_4\in\mathcal{I}\\ k_3>j_2>k_2+1,k_1>j_4}} \epsilon'(k_3+1,k_2)\epsilon'(j_2,j_4) - \sum_{\substack{j_1,j_3\in\mathcal{I}\\ k_3>j_1>k_2+1,k_2>j_3>k_1+1}} \epsilon'(j_1,j_3)\epsilon'(k_2+1,k_1)\\
&- \epsilon'(k_2+1,k_1)\epsilon'(k_3+1,k_2)\\
&- \epsilon'(k_3+1,k_2)\epsilon'(k_3,k_1+1)\\
&- \epsilon'(k_3,k_1+1)\epsilon'(k_2+1,k_1).
\end{aligned}
\tag{4.20}
$$

Using the relations (3.7) between ϵ and ϵ' for the Reidemeister-III move it is easy to see that most of these terms cancel, and we are left with

$$
\begin{aligned}
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = \; &- \; \epsilon'(k_2+1,k_1)\epsilon'(k_3+1,k_2)\\
&- \; \epsilon'(k_3+1,k_2)\epsilon'(k_3,k_1+1)\\
&- \; \epsilon'(k_3,k_1+1)\epsilon'(k_2+1,k_1) = +1,
\end{aligned}
\tag{4.21}
$$

due to the cyclic relation (3.6) for ϵ'.

5 The limit of flat knots

We shall now define the limit of flat knots. As projection space $\mathbb{R}^2 \times \{0\} = \{(x,y,0) \in \mathbb{R}^3\}$ will now be used. Consider some knot diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$; the shadow diagram is $K = \text{image}(\pi)$, the set of crossings is $\mathcal{C} \subset \mathbb{R}^2$. We will make some assumptions for simplicity, but without loss of generality. The first of them is formulated as

$$|\dot{\pi}(s)| = \textit{const} \;\text{ for all }\; s \in I_{/\sim}. \tag{5.1}$$

For all $c \in \mathcal{C}$ let $U_c \subset \mathbb{R}^2$ be a sufficiently small, open disk with center c. Furthermore, we define

$$U = \mathbb{R}^2 \setminus \bigcup_{c\in\mathcal{C}} U_c. \tag{5.2}$$

Sufficiently small in the previous definition means that only one crossing is contained in every U_c and the basepoint $\pi(0) = \pi(1)$ lies outside every U_c. The projection space is a deformation retract of $\mathbb{R}^3$ with respect to the homotopy

$$H_t : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \mapsto (x,y,tz), \;\; t \in [0,1]. \tag{5.3}$$

Hence H_0 projects the whole space onto the projection space and H_1 is the identity in $\mathbb{R}^3$. Now we consider a knot $K_0 \subset \mathbb{R}^3$ parametrized by $\pi_0 : I_{/\sim} \to \mathbb{R}^3$ with the following properties:

(i) $H_0 \circ \pi_0 = \pi$, therefore $H_0(K_0) = K$.

(ii) For every $s \in I_{/\sim}$ with $\pi_0(s) \in U_c \times \mathbb{R}$ the third component of $\dot{\pi}_0(s)$ vanishes and $\ddot{\pi}_0(s) = 0$, this means that the knot line is straight in this region.

Every cylinder $U_c \times \mathbb{R}$ is crossed by two straight lines. They will be called g_c^+ and g_c^- so that

$$s_+ > s_- \text{ for all } s_+ \in \pi_0^{-1}(g_c^+) \text{ and } s_- \in \pi_0^{-1}(g_c^-). \tag{5.4}$$

We introduce another condition for convenience, namely $H_0(g_c^+) \perp H_0(g_c^-)$. The situation is shown in figure **??**. Consider a diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ and an analytically formulated ambient isotopy invariant $f(K_0)$ which is based on the evaluation of an expression defined using the parametrization π_0 with $K_0 = \text{image}(\pi_0)$ and $\pi = H_0 \circ \pi_0$. Then

$$f(K_0) = \lim_{t \to 0} f(H_t(K_0)) =: f(\mathcal{K}) \tag{5.5}$$

since H_t is an ambient isotopy for every $t \in]0, 1]$. This limit will be used for calculating the line integral ρ_2.

6 Calculation of $\rho_2(K_0)$ for the limit of flat knots

In this section we calculate $\rho_2(K_0)$ for the limit of flat knots. For any knot diagram $\mathcal{K}$ a knot K_0 constructed as in section 5 may be given. The integral ρ_2 for this knot is given by

$$\begin{aligned} \rho_2(K_0) &= \frac{1}{8\pi^2} \int_{K_0} dx_1^{\mu_1} \int_{\mathrm{BP}}^{x_1} dx_2^{\mu_2} \int_{\mathrm{BP}}^{x_2} dx_3^{\mu_3} \int_{\mathrm{BP}}^{x_3} dx_4^{\mu_4} \;\; \epsilon_{\mu_4\mu_2\sigma_2}\epsilon_{\mu_3\mu_1\sigma_1} \\ &\times \frac{(x_4 - x_2)^{\sigma_2}}{|x_4 - x_2|^3} \frac{(x_3 - x_1)^{\sigma_1}}{|x_3 - x_1|^3}, \end{aligned} \tag{6.1}$$

where BP denotes the basepoint. Using a parametrization

$$x : I_{/\sim} \to \mathbb{R}^3 \tag{6.2}$$

it is written as

$$\begin{aligned} \rho_2(K_0) &= \frac{1}{8\pi^2} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_3} ds_4 \;\; \dot{x}(s_1)^{\mu_1} \dot{x}(s_2)^{\mu_2} \dot{x}(s_3)^{\mu_3} \dot{x}(s_4)^{\mu_4} \\ &\times \;\; \epsilon_{\mu_4\mu_2\sigma_2}\epsilon_{\mu_3\mu_1\sigma_1} \frac{(x(s_4) - x(s_2))^{\sigma_2}}{|x(s_4) - x(s_2)|^3} \frac{(x(s_3) - x(s_1))^{\sigma_1}}{|x(s_3) - x(s_1)|^3}. \end{aligned} \tag{6.3}$$

The total integration range is therefore

$$\Delta_4 := \{(s_1, s_2, s_3, s_4) | 1 > s_1 > s_2 > s_3 > s_4 \geq 0\}, \tag{6.4}$$

and it will be divided into several parts, classified by the location of the $x(s_i)$ with respect to the crossings. For every crossing c we define intervals I_c^+ and I_c^- in $I_{/\sim}$ with the following properties (cf. figure **??**):

$$x(I_c^+) = g_c^+ \text{ and } x(I_c^-) = g_c^-. \tag{6.5}$$

Due to the construction of the U_c, $\pi(0)$ is not contained in any of the $I_c^{+/-}$ and all intervals $I_c^{+/-}$ are disjoint. The following cases are considered:

1. $x(s_1), x(s_3) \in U_c \times \mathbb{R}$, with $x(s_1) \in g_c^+$ and $x(s_3) \in g_c^-$,
   $x(s_2), x(s_4) \in U_d \times \mathbb{R}$, with $x(s_2) \in g_d^+$ and $x(s_4) \in g_d^-$,
   i.e. two pairs of integration variables meet at different crossings c and d (see figure **??**). This part will be denoted by $\Delta_4^{(1)}(c, d)$ and is defined as

   $$\Delta_4^{(1)}(c, d) = \{(s_1, s_2, s_3, s_4) \in I_c^+ \times I_d^+ \times I_c^- \times I_d^- \mid s_1 > s_2 > s_3 > s_4\}. \tag{6.6}$$

   Note that this set can be empty for certain choices of c and d.

2. $x(s_1), x(s_3) \in U_c \times \mathbb{R}$, with $x(s_1) \in g_c^+$ and $x(s_3) \in g_c^-$,
   $x(s_2), x(s_4) \in U_c \times \mathbb{R}$, with $x(s_2) \in g_c^+$ and $x(s_4) \in g_c^-$,
   i.e. all integration variables meet at the same crossing c in the way shown in figure **??**. This part will be denoted by $\Delta_4^{(2)}(c)$ and is defined as

   $$\Delta_4^{(2)}(c) = \{(s_1, s_2, s_3, s_4) \in I_c^+ \times I_c^+ \times I_c^- \times I_c^- \mid s_1 > s_2 > s_3 > s_4\}. \tag{6.7}$$

3. Other cases, which do not contribute to ρ_2. This part will be called Δ_4^{Rest}. These cases contain at least one pair of integration variables $(x(s_1), x(s_3))$ or $(x(s_2), x(s_4))$ which do not encounter at any crossing in the way described in the previous cases.

The total integration range can then be written as

$$\Delta_4 = \bigcup_{c,d \in \mathcal{C}} \Delta_4^{(1)}(c, d) \;\cup\; \bigcup_{c \in \mathcal{C}} \Delta_4^{(2)}(c) \;\cup\; \Delta_4^{\text{Rest}}, \tag{6.8}$$

and the integral over K_0 is written as

$$\begin{aligned} \rho_2(K_0) &=: \rho_2(K_0; \Delta_4) \\ &= \sum_{c,d \in \mathcal{C}} \rho_2(K_0; \Delta_4^{(1)}(c, d)) + \sum_{c \in \mathcal{C}} \rho_2(K_0; \Delta_4^{(2)}(c)) + \rho_2(K_0; \Delta_4^{\text{Rest}}), \end{aligned} \tag{6.9}$$

where the second argument of ρ_2 denotes the respective integration range.

Case 1: Four variables encounter at two crossings. The corresponding integration range containing two crossings c and d is $\Delta_4^{(1)}(c,d)$. The integral $\rho_2(K_0;\Delta_4^{(1)}(c,d))$ can be split into two factors:

$$\begin{aligned}\rho_2(K_0;\Delta_4^{(1)}(c,d)) \;=\; & \frac{1}{8\pi^2}\int_{I_c^+} ds_1 \int_{I_c^-} ds_3 \;\; \dot{x}(s_1)^{\mu_1}\dot{x}(s_3)^{\mu_3}\epsilon_{\mu_3\mu_1\sigma_1}\frac{(x(s_3)-x(s_1))^{\sigma_1}}{|x(s_3)-x(s_1)|^3} \\ & \times \int_{I_d^+} ds_2 \int_{I_d^-} ds_4 \;\; \dot{x}(s_2)^{\mu_2}\dot{x}(s_4)^{\mu_4}\epsilon_{\mu_4\mu_2\sigma_2}\frac{(x(s_4)-x(s_2))^{\sigma_2}}{|x(s_4)-x(s_2)|^3}.\end{aligned} \tag{6.10}$$

We now calculate one of these factors for the limit of flat knots. The expression

$$\int_{I_c^+} ds_1 \int_{I_c^-} ds_3 \;\; \dot{x}(s_1)^{\mu_1}\dot{x}(s_3)^{\mu_3}\epsilon_{\mu_3\mu_1\sigma_1}\frac{(x(s_3)-x(s_1))^{\sigma_1}}{|x(s_3)-x(s_1)|^3} \tag{6.11}$$

is reparametrizable. We use parameters $s_1', s_3' \in \,]-1,+1[$ and a corresponding parametrization $x_1', x_3' : \,]-1,+1[\, \rightarrow \mathbb{R}^3$. The expression (6.11) is invariant with respect to scaling, rotating, and translating the coordinate frame. So the coordinates can be chosen in such a way that

$$x_1'(s_1')^{\mu_1} = (s_1', 0, z_1) \quad \text{and} \quad x_3'(s_3')^{\mu_3} = (0, s_3', z_3) \tag{6.12}$$

and hence

$$\dot{x}_1'(s_1')^{\mu_1} = (1,0,0) \quad \text{and} \quad \dot{x}_3'(s_3')^{\mu_3} = (0,1,0). \tag{6.13}$$

After defining $h = z_1 - z_3$ the crossing information of c is simply

$$\epsilon(c) = \mathrm{sgn}(h). \tag{6.14}$$

The evaluation of the vector products gives

$$\int_{-1}^{+1} ds_1' \int_{-1}^{+1} ds_3' \;\; \frac{h}{\sqrt{s_1'^2 + s_3'^2 + h^2}^{\,3}}. \tag{6.15}$$

The limit of a flat knot is now equivalent to the limit $h \rightarrow 0$. In this limit the integral assumes the value

$$2\pi\epsilon(c), \tag{6.16}$$

and therefore

$$\rho_2(K_0;\Delta_4^{(1)}(c,d)) = \frac{1}{2}\epsilon(c)\epsilon(d) \tag{6.17}$$

if for the crossings c and d the range $\Delta_4^{(1)}(c,d)$ is not empty.

Case 2: Four variables encounter at one crossing. The integration range is $\Delta_4^{(2)}(c)$. For this range the integral cannot be factorized as in the previous case.

$$\begin{aligned}\rho_2(K_0;\Delta_4^{(2)}(c)) &= \frac{1}{8\pi^2}\int_{I_c^+\times I_c^+,s_1>s_2} ds_1\, ds_2 \int_{I_c^-\times I_c^-,s_3>s_4} ds_3\, ds_4\, \dot{x}(s_1)^{\mu_1}\dot{x}(s_2)^{\mu_2}\dot{x}(s_3)^{\mu_3}\dot{x}(s_4)^{\mu_4} \\ &\times\ \epsilon_{\mu_4\mu_2\sigma_2}\epsilon_{\mu_3\mu_1\sigma_1}\frac{(x(s_4)-x(s_2))^{\sigma_2}}{|x(s_4)-x(s_2)|^3}\frac{(x(s_3)-x(s_1))^{\sigma_1}}{|x(s_3)-x(s_1)|^3}.\end{aligned} \tag{6.18}$$

Again, as in case 1, the parametrizations x'_1 and x'_3 are used, with parameters in the range $]-1,+1[$. Under the reparametrization we replace

$$\begin{array}{lcl} x(s_1)\to x'_1(s'_1) & \text{and} & x(s_2)\to x'_1(s'_2) \\ x(s_3)\to x'_3(s'_3) & \text{and} & x(s_4)\to x'_3(s'_4) \end{array} \tag{6.19}$$

since s_1 and s_2 relate to the same piece g_c^+, and s_3 and s_4 relate to g_c^-. We then get

$$\begin{aligned}&\frac{1}{8\pi^2}\int_{-1}^{+1} ds'_1\int_{-1}^{s'_1} ds'_2\int_{-1}^{+1} ds'_3\int_{-1}^{s'_3} ds'_4\ \dot{x}'_1(s'_1)^{\mu_1}\dot{x}'_1(s'_2)^{\mu_2}\dot{x}'_3(s'_3)^{\mu_3}\dot{x}'_3(s'_4)^{\mu_4} \\ \times\ &\epsilon_{\mu_4\mu_2\sigma_2}\epsilon_{\mu_3\mu_1\sigma_1}\frac{(x'_3(s'_3)-x'_1(s'_1))^{\sigma_1}}{|x'_3(s'_3)-x'_1(s'_1)|^3}\frac{(x'_3(s'_4)-x'_1(s'_2))^{\sigma_2}}{|x'_3(s'_4)-x'_1(s'_2)|^3}\end{aligned} \tag{6.20}$$

and after evaluating the vector products and using the flat knot limit:

$$\lim_{h\to 0}\frac{1}{8\pi^2}\int_{-1}^{+1} ds'_1\int_{-1}^{s'_1} ds'_2\int_{-1}^{+1} ds'_3\int_{-1}^{s'_3} ds'_4\ \frac{h}{\sqrt{s_1'^2+s_3'^2+h^2}^3}\frac{h}{\sqrt{s_2'^2+s_4'^2+h^2}^3}=\frac{1}{8}. \tag{6.21}$$

So the integral $\rho_2(K_0;\Delta_4^{(2)}(c))$ for any crossing $c\in\mathcal{C}$ is

$$\rho_2(K_0;\Delta_4^{(2)}(c))=\frac{1}{8}. \tag{6.22}$$

Case 3: At least one pair of variables does not encounter at any crossing. We assume that $x(s_1)$ and $x(s_3)$ lie in different parts of the knot, e.g. in $U\times\mathbb{R}$ and some $U_c\times\mathbb{R}$, or in cylinders associated to different crossings. Then the triple product

$$\dot{x}(s_1)^{\mu_1}\dot{x}(s_3)^{\mu_3}\epsilon_{\mu_3\mu_1\sigma_1}\frac{(x(s_3)-x(s_1))^{\sigma_1}}{|x(s_3)-x(s_1)|^3} \tag{6.23}$$

and therefore the whole integral in this range vanishes in the limit of flat knots. If $x(s_1)$ and $x(s_3)$ lie in the same cylinder or in $U\times\mathbb{R}$, and $x(s_3)$ lies near $x(s_1)$ the expression (6.23) vanishes as well, as can be shown by expanding $x(s_3)$ in a power series near $x(s_1)$.

Using n, the self-crossing number χ_2 from section 4.1 and the results (6.17) and (6.22) $\rho_2(\mathcal{K})$ assumes the following form

$$\rho_2(\mathcal{K}):=\lim_{t\to 0}\rho_2(H_t(K_0))=\frac{1}{2}\chi_2(\mathcal{K})+\frac{1}{8}n. \tag{6.24}$$

7 Construction of $\rho_1(\mathcal{K})$ and $\rho^{\mathrm{II}}(\mathcal{K})$

The behaviour of $\rho_2(\mathcal{K})$ under the Reidemeister moves follows from the results of section 4.3:

$$\rho_2(L_{\mathrm{I}}^{+/-}) - \rho_2(L_{\mathrm{I}}^0) = +\frac{1}{8} \qquad \rho_2(L_{\mathrm{II-A}}) - \rho_2(L_{\mathrm{II-A}}^0) = -\frac{1}{4}$$

$$\rho_2(L_{\mathrm{II-B}}) - \rho_2(L_{\mathrm{II-B}}^0) = +\frac{1}{4} \qquad \rho_2(L_{\mathrm{III}}^+) - \rho_2(L_{\mathrm{III}}^-) = +\frac{1}{2}. \tag{7.1}$$

Furthermore, we know that $\rho_2(U_0) = 0$, where U_0 is the unknot diagram without crossing. We want to construct an ambient isotopy invariant

$$\rho^{\mathrm{II}}(\mathcal{K}) = \rho_1(\mathcal{K}) + \rho_2(\mathcal{K}), \tag{7.2}$$

i.e. an object based on the evaluation of $\mathcal{K}$, invariant under the Reidemeister moves. Therefore, we have to postulate a behaviour of ρ_1 opposite to that of ρ_2:

$$\rho_1(L_{\mathrm{I}}^{+/-}) - \rho_1(L_{\mathrm{I}}^0) = -\frac{1}{8} \qquad \rho_1(L_{\mathrm{II-A}}) - \rho_1(L_{\mathrm{II-A}}^0) = +\frac{1}{4}$$

$$\rho_1(L_{\mathrm{II-B}}) - \rho_1(L_{\mathrm{II-B}}^0) = -\frac{1}{4} \qquad \rho_1(L_{\mathrm{III}}^+) - \rho_1(L_{\mathrm{III}}^-) = -\frac{1}{2}. \tag{7.3}$$

From an analytical calculation in [4] we know that

$$\rho_1(U_0) = -\frac{1}{12}, \tag{7.4}$$

if U_0 represents an unknot that lies within a plane. In the appendix it will be shown that in the limit of flat knots ρ_1 is independent of the values of the crossing function ϵ. This means that for any two diagrams $\mathcal{K}$ and $\mathcal{K}'$ which differ only in the crossing functions one has

$$\rho_1(\mathcal{K}) = \rho_1(\mathcal{K}'). \tag{7.5}$$

Therefore, instead of calculating ρ_1 for $\mathcal{K}$ we can use the standard ascending diagram $\alpha(\mathcal{K})$ with respect to the basepoint:

$$\rho_1(\mathcal{K}) = \rho_1(\alpha(\mathcal{K})). \tag{7.6}$$

For any diagram $\mathcal{K}$ the standard ascending diagram $\alpha(\mathcal{K})$, as defined in [5], is obtained by passing through the knot, starting from the basepoint, and switching each crossing encountered for the first time to an undercrossing. A standard ascending diagram is a diagram representation of the unknot.

It should be possible to show directly that the analytical expression for ρ_1 fulfils the above conditions in the limit of flat knots. Nevertheless, the properties (7.3), (7.4), and (7.6) suffice to construct ρ_1 in a well-defined way and to show that it is unique. For proving the uniqueness we assume that there is another expression $\tilde{\rho}_1$ with the same properties. Let

$\mathcal{K}$ be a knot diagram. Then $\alpha(\mathcal{K})$ can be obtained from U_0 by applying a finite sequence of Reidemeister moves $R_1, \ldots, R_m$:

$$\alpha(\mathcal{K}) = R_m \ldots R_1 \ U_0. \tag{7.7}$$

Since $\tilde{\rho}_1(U_0) = \rho_1(U_0)$ and the change of both under Reidemeister moves is the same, also the result is the same. Hence

$$\tilde{\rho}_1(\mathcal{K}) = \tilde{\rho}_1(\alpha(\mathcal{K})) = \rho_1(\alpha(\mathcal{K})) = \rho_1(\mathcal{K}). \tag{7.8}$$

We now demonstrate that ρ_1 is well-defined. Again, let $\mathcal{K}$ be a knot which can be obtained from U_0 in two different ways, i.e. by applying a sequence $R_1, \ldots, R_m$ or another sequence $R'_1, \ldots, R'_{m'}$. In principle one has to show that both sequences for ρ_1 lead to the same result. One can, however, simply define

$$\rho_1(\mathcal{K}) = -\frac{1}{12} - \rho_2(\alpha(\mathcal{K})) \tag{7.9}$$

and check that the defining conditions are fulfilled. For condition (7.4) this is clear. For condition (7.6) it is clear as well, since

$$\alpha(\alpha(\mathcal{K})) = \alpha(\mathcal{K}). \tag{7.10}$$

For the properties (7.3) some remarks have to be made. Consider a knot $\mathcal{K}$ which contains a Reidemeister situation L and another one $\mathcal{K}'$ differing from $\mathcal{K}$ by containing a Reidemeister situation L^0 instead of L. Then the move can be applied to $\mathcal{K}$. However, if the basepoint is chosen in an *inappropriate* way, e.g. between k_1 and k_1+1 or between k_2 and k_2+1 in $L_{\mathrm{II-A}}$ of figure **??**, the move cannot transform $\alpha(\mathcal{K})$ into $\alpha(\mathcal{K}')$. Therefore we will place the basepoint in such a way that the move can be applied, i.e. *outside* the Reidemeister situation and prove the independence of ρ_1 of the choice of the basepoint afterwards. If the move can be applied, it is clear by construction that ρ_1 behaves correctly.

Proposition. *$\rho_2(\alpha(\mathcal{K}))$ is independent of the choice of the basepoint.*

Proof. It is sufficient to show that the basepoint can be shifted by one crossing without changing ρ_2. Since the number of crossings n is invariant under this operation, it is sufficient to show the invariance of χ_2.

Consider a diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$. The basepoint b is a point in the piece between two indices $i_b - 1$ and i_b. The basepoint shifted by one crossing lies in the piece between i_b and $i_b + 1$. Let $\alpha(\mathcal{K}) = (\pi, \epsilon_\alpha, \mathcal{I})$ and $\alpha'(\mathcal{K}) = (\pi, \epsilon_{\alpha'}, \mathcal{I})$ denote the standard ascending diagrams of $\mathcal{K}$ with respect to the basepoints b and b'. From the definition of the standard ascending diagrams it is clear that the difference between $\alpha(\mathcal{K})$ and $\alpha'(\mathcal{K})$ lies in the crossing functions at i_b:

$$\epsilon_\alpha(i_b) = -\epsilon_{\alpha'}(i_b). \tag{7.11}$$

The index which forms a crossing together with i_b will be denoted by $\bar{i_b} \in \mathcal{I}$. We have to show that

$$\chi_2(\alpha(\mathcal{K})) - \chi_2(\alpha'(\mathcal{K})) = 0. \tag{7.12}$$

In section 4.2 it has been shown that for two diagrams which differ only in one crossing (in this case $\alpha(\mathcal{K})$ and $\alpha'(\mathcal{K})$ differ in $\pi(s_{i_b})$), the difference for χ_2 is

$$\chi_2(\alpha(\mathcal{K})) - \chi_2(\alpha'(\mathcal{K})) = 2\epsilon_\alpha(i_b, \bar{i_b})\chi_1(\,]i_b, \bar{i_b}[\,,\,]\bar{i_b}, i_b[\,;\epsilon_\alpha). \tag{7.13}$$

The second factor is related to the linking number between the two link components corresponding to the pieces $L^1 = \pi(\,]s_{i_b}, s_{\bar{i_b}}[\,)$ and $L^0 = \pi(\,]s_{\bar{i_b}}, s_{i_b}[\,)$ according to equation (4.5) . Since $\alpha(\mathcal{K})$ is an ascending diagram, the component L^1 lies completely underneath L^0, which means that they can be separated in space. Hence their linking number is zero. This completes the proof.

Finally, the invariant from the second order term of the Chern-Simons theory assumes the form

$$\begin{aligned}\rho^{\mathrm{II}}(\mathcal{K}) &= \rho_1(\mathcal{K}) + \rho_2(\mathcal{K}) \\ &= -\frac{1}{12} + \frac{1}{2}\{\chi_2(\mathcal{K}) - \chi_2(\alpha(\mathcal{K}))\}.\end{aligned} \tag{7.14}$$

This formula is the main result of the present paper.

8 Relation between ρ^{II} and the total twist

We define

$$\tau(\mathcal{K}) = \frac{1}{2}\left(\rho^{\mathrm{II}}(\mathcal{K})) + \frac{1}{12}\right) = \frac{1}{4}\{\chi_2(\mathcal{K}) - \chi_2(\alpha(\mathcal{K}))\}. \tag{8.1}$$

The aim of this section is to show that this is equivalent to the total twist which was defined by Lickorish and Millet in [5]. Consider some knot diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ and its standard ascending diagram $\alpha(\mathcal{K}) = (\pi, \epsilon_\alpha, \mathcal{I})$, both of course with the same shadow diagram and the same set of crossings $\mathcal{C}$. The difference between the two diagrams may consist in m crossings $c_1, \ldots, c_m \in \mathcal{C}$ such that $\epsilon(c_i) = -\epsilon_\alpha(c_i)$ for $i = 1 \ldots m$. We shall here use a notation similar to that used in [5]. Let $\sigma_i, i = 1 \ldots m$ be the operation applied to a diagram which switches the crossing c_i. Now define $\mathcal{K}_0 = \alpha(\mathcal{K})$ and $\mathcal{K}_j = (\pi, \epsilon_j, \mathcal{I})$ with

$$\mathcal{K}_j = \sigma_j\sigma_{j-1}\ldots\sigma_1\mathcal{K}_0 \tag{8.2}$$

so that $\mathcal{K} = \mathcal{K}_m$. Then τ can be written as

$$\tau(\mathcal{K}) = \frac{1}{4}\{\chi_2(\mathcal{K}_m) - \chi_2(\mathcal{K}_0)\}. \tag{8.3}$$

For every $\mathcal{K}_j$ and every $c \in \mathcal{C}$ formed by the indices i_c^+ and i_c^- we define the non-overlapping pieces $S_j^+(c) = \pi(\,]s_{i_c^+}, s_{i_c^-}[\,)$ and $S_j^-(c) = \pi(\,]s_{i_c^-}, s_{i_c^+}[\,)$ and the link components $L_j^+(c)$

and $L_j^-(c)$, which arise from nullifying the crossing c. The two slightly different notations are used to draw the connection between our formulation and the one in [5]. In section 4.2 we have already calculated the change of χ_2 under the change of one crossing, or in this case the change under the operation σ_j. The result can be written as

$$\chi_2(\mathcal{K}_j) - \chi_2(\mathcal{K}_{j-1}) = 2\epsilon_j(c_j)\chi_1(S_j^+(c_j), S_j^-(c_j)). \tag{8.4}$$

Using relation (8.4) we write $\tau(\mathcal{K})$ as

$$\begin{aligned} \tau(\mathcal{K}) &= \frac{1}{4}\{\chi_2(\mathcal{K}_m) - \chi_2(\mathcal{K}_{m-1}) + \chi_2(\mathcal{K}_{m-1}) - \chi_2(\mathcal{K}_{m-2}) + \ldots + \chi_2(\mathcal{K}_1) - \chi_2(\mathcal{K}_0)\} \\ &= \frac{1}{2}\sum_{j=1}^{m} \epsilon_j(c_j)\chi_1(S_j^+(c_j), S_j^-(c_j)). \end{aligned} \tag{8.5}$$

The link components $L_j^+(c_j)$ and $L_j^-(c_j)$, which correspond to the pieces $S_j^+(c_j)$ and $S_j^-(c_j)$ in the previous equation, coincide precisely with the definitions of L_j^1 and L_j^0 used in [5]. Therefore, using $\epsilon_j(c_j) = \epsilon(c_j)$, the definitions of L_j^1 and L_j^0, and the relation (4.5) between λ and χ_1 one obtains

$$\tau(\mathcal{K}) = \sum_{j=1}^{m} \epsilon(c_j)\lambda(L_j^1, L_j^0), \tag{8.6}$$

which is the same expression as the one given in [5].

9 Example: Calculation of ρ^{II}

Finally, we shall demonstrate the calculation of ρ^{II} for a specific knot. We choose the knot 5_2 in the notation of Rolfsen in [9], shown below in figure **??**. The *calculation* consists of listing all possible non-vanishing contributions in the fourfold sums χ_2 with crossing function ϵ and its standard ascending version with crossing function ϵ_α. In table 1 below all significant combinations of indices for two variables j_1, j_3 or j_2, j_4 fulfilling $j_1 > j_3$ and $j_2 > j_4$ are listed, together with the corresponding values of ϵ and ϵ_α. From this table all non-vanishing contributions for $j_1 > j_2 > j_3 > j_4$ are constructed (see table 2). The results are

$$\chi_2(5_2) = +7 \quad \text{and} \quad \chi_2(\alpha(5_2)) = -1, \tag{9.1}$$

and therefore

$$\rho^{\mathrm{II}}(5_2) = \frac{47}{12} \quad \text{and} \quad \tau(5_2) = 2. \tag{9.2}$$

j_1	j_3	$\epsilon(j_1, j_3)$	$\epsilon_\alpha(j_1, j_3)$
j_2	j_4	$\epsilon(j_2, j_4)$	$\epsilon_\alpha(j_2, j_4)$
5	0	$-$	$+$
6	1	$-$	$-$
7	4	$-$	$+$
8	3	$-$	$-$
9	2	$-$	$+$

Table 1: Encounters of two variables in the knot 5_2.

j_1	j_2	j_3	j_4	$\epsilon(j_1, j_3)\epsilon(j_2, j_4)$	$\epsilon_\alpha(j_1, j_3)\epsilon_\alpha(j_2, j_4)$
6	5	1	0	$+$	$-$
7	5	4	0	$+$	$+$
7	6	4	1	$+$	$-$
8	5	3	0	$+$	$-$
8	6	3	1	$+$	$+$
9	5	2	0	$+$	$+$
9	6	2	1	$+$	$-$
				$+7$	-1

Table 2: All non-vanishing contributions to $\chi_2(\mathcal{K})$ and $\chi_2(\alpha(\mathcal{K}))$.

10 Outlook

As already emphasized, the method of flattening the knot in order to calculate the complicated line integrals from the perturbative expansion of the Wilson loops can be applied to higher orders as well. We have already checked this by the use of a computer program in C++ to be run on a PC which calculates $\rho^{\mathrm{II}}(\mathcal{K})$ and the third order invariant $\rho^{\mathrm{III}}(\mathcal{K})$ for arbitrary knots. Whoever is interested in this program may order it via e-mail. We shall soon publish a discussion of the third order calculation which employs the formalism developed in this paper.

One may hope that a systematic examination of the higher orders will yield a convenient description of the Vassiliev invariants, which are conjectured to classify the knots uniquely [6]. The computation time for the invariants ρ^{II} and ρ^{III} using the procedure presented here grows as $O(n^2)$ and $O(n^3)$ respectively. This is to be compared to computations involving polynomial invariants, whose complexity grows exponentially with crossing number [7].

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Appendix
Independence of ρ_1 with respect to the values of ϵ

It will now be demonstrated that $\rho_1(\mathcal{K})$ is not dependent on the values of the crossing information ϵ, i.e. that ρ_1 can be interpreted as a functional of the shadow diagram. The analytical form of ρ_1 for a knot K_0 is given by [4]

$$\rho_1(K_0) = -\frac{1}{32\pi^3}\int_{K_0} dx_1^{\mu_1}\int_{\mathrm{BP}}^{x_1} dx_2^{\mu_2}\int_{\mathrm{BP}}^{x_2} dx_3^{\mu_3}\ \epsilon^{\nu_1\nu_2\nu_3}\epsilon_{\mu_1\nu_1\sigma_1}\epsilon_{\mu_2\nu_2\sigma_2}\epsilon_{\mu_3\nu_3\sigma_3}I^{\sigma_1\sigma_2\sigma_3}(x_1,x_2,x_3) \tag{A.1}$$

with

$$I^{\sigma_1\sigma_2\sigma_3}(x_1,x_2,x_3) = \int d^3z\frac{(z-x_1)^{\sigma_1}}{|z-x_1|^3}\frac{(z-x_2)^{\sigma_2}}{|z-x_2|^3}\frac{(z-x_3)^{\sigma_3}}{|z-x_3|^3}. \tag{A.2}$$

It was also demonstrated in [4] that this integral can be solved and assumes the form

$$I^{\sigma_1\sigma_2\sigma_3}(x_1,x_2,x_3) = -\frac{\partial}{\partial x_2^{\sigma_2}}\frac{\partial}{\partial x_3^{\sigma_3}}I^{\sigma_1}(x_2-x_1,x_1-x_3) \tag{A.3}$$

with

$$I^{\sigma_1}(c,b) = 2\pi\frac{|c|+|b|-|c+b|}{|c||b|-c\cdot b}\left\{\frac{c^{\sigma_1}}{|c|}-\frac{b^{\sigma_1}}{|b|}\right\} \tag{A.4}$$

where a slightly different notation has been used in comparison to [4] in order to formulate the expression in a symmetric way. If we define

$$a = x_3 - x_2, \quad b = x_1 - x_3, \quad c = x_2 - x_1 \tag{A.5}$$

so that $a = -b-c$, the expression I^{σ_1} can be written as

$$\begin{aligned} I^{\sigma_1}(b,c) &= 4\pi\frac{1}{|a|+|b|+|c|}\left\{\frac{c^{\sigma_1}}{|c|}-\frac{b^{\sigma_1}}{|b|}\right\} \\ &= \left\{\frac{\partial}{\partial c^{\sigma_1}}-\frac{\partial}{\partial b^{\sigma_1}}\right\}4\pi\ln(|a|+|b|+|c|). \end{aligned} \tag{A.6}$$

a, b, c will be considered as functions of x_1, x_2, x_3 in the following. The partial derivatives in (A.3) can be reformulated as partial derivatives with respect to a, b, and c. Then $I^{\sigma_1\sigma_2\sigma_3}$ is formulated in a symmetric way:

$$I^{\sigma_1\sigma_2\sigma_3}(x_1,x_2,x_3) = \left\{\frac{\partial}{\partial b^{\sigma_1}}-\frac{\partial}{\partial c^{\sigma_1}}\right\}\left\{\frac{\partial}{\partial c^{\sigma_2}}-\frac{\partial}{\partial a^{\sigma_2}}\right\}\left\{\frac{\partial}{\partial a^{\sigma_3}}-\frac{\partial}{\partial b^{\sigma_3}}\right\}4\pi\ln(|a|+|b|+|c|). \tag{A.7}$$

This can be calculated straightforwardly. We use the abbreviations

$$N = |a|+|b|+|c|, \quad m^\sigma(u,v) = \frac{u^\sigma}{|u|}-\frac{v^\sigma}{|v|}, \quad m^{\sigma\tau}(u) = \frac{1}{|u|}\delta^{\sigma\tau}-\frac{u^\sigma u^\tau}{|u|^3} \tag{A.8}$$

and obtain

$$\begin{aligned} I^{\sigma_1\sigma_2\sigma_3}(x_1,x_2,x_3) &= 4\pi\left(\frac{2}{N^3}m^{\sigma_1}(b,c)m^{\sigma_2}(c,a)m^{\sigma_3}(a,b)\right. \\ &+ \left.\frac{1}{N^2}\left(m^{\sigma_1\sigma_3}(b)m^{\sigma_2}(c,a)+m^{\sigma_3\sigma_2}(a)m^{\sigma_1}(b,c)+m^{\sigma_2\sigma_1}(c)m^{\sigma_3}(a,b)\right)\right). \end{aligned} \tag{A.9}$$

Using a parametrization $x(s)$ and the abbreviations $x_i := x(s_i)$, $\dot{x}_i := \dot{x}(s_i)$, $\hat{u}$ for the unit vector of any vector u, the line integral ρ_1 assumes the form

$$\begin{aligned} \rho_1(K_0) = \frac{1}{8\pi^2}\int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \quad & \frac{2}{N^3}\left[\dot{x}_1 \times m(b,c)\right]\cdot\left[\dot{x}_2 \times m(c,a)\right]\times\left[\dot{x}_3 \times m(a,b)\right] \\ + \; & \frac{1}{N^2}\left\{\dot{x}_1 \times m(b,c)\cdot\frac{1}{|a|}(\dot{x}_2\times\dot{x}_3 - \hat{a}(\dot{x}_2\times\dot{x}_3\cdot\hat{a}))\right. \\ & \quad + \; \dot{x}_2 \times m(c,a)\cdot\frac{1}{|b|}(\dot{x}_3\times\dot{x}_1 - \hat{b}(\dot{x}_3\times\dot{x}_1\cdot\hat{b})) \\ & \quad + \left.\; \dot{x}_3 \times m(a,b)\cdot\frac{1}{|c|}(\dot{x}_1\times\dot{x}_2 - \hat{c}(\dot{x}_1\times\dot{x}_2\cdot\hat{c}))\right\}. \end{aligned} \tag{A.10}$$

Some properties of the integrand will now be examined. We shall refer to it as $h(x_1,x_2,x_3) = h(x(s_1),x(s_2),x(s_3))$ so that

$$\rho_1(K_0;\Delta_3) := \rho_1(K_0) = \int_{\Delta_3} ds_1\, ds_2\, ds_3\;\; h(x(s_1),x(s_2),x(s_3)), \tag{A.11}$$

where

$$\Delta_3 = \{(s_1,s_2,s_3)\mid 1 > s_1 > s_2 > s_3 \geq 0\}. \tag{A.12}$$

The property of $h(x_1,x_2,x_3)$ which is important here is its invariance under parity transformations. As a consequence

$$(\lim_{\substack{t\to 0\\ t>0}} - \lim_{\substack{t\to 0\\ t<0}})h(H_t\circ x(s_1), H_t\circ x(s_2), H_t\circ x(s_3)) = 0, \tag{A.13}$$

where the function H_t from section 5 has been used, now however with $t \in \mathbb{R}$. The integration range can be split into three subsets.

1. $x(s_1), x(s_2), x(s_3) \in U_c \times \mathbb{R}$ and at least one of the $x(s_i)$ is in g_c^+ and another in g_c^-. This part will be denoted as $\Delta_3^{(1)}(c)$.

2. $x(s_2), x(s_3) \in U_c \times \mathbb{R}$, $x(s_1) \notin U_c \times \mathbb{R}$, with $x(s_2) \in g_c^+$ and $x(s_3) \in g_c^-$, and further permutations. These parts together will be denoted as $\Delta_3^{(2)}(c)$.

3. Other cases, where no two variables cross each other. These situations are contained in Δ_3^{Rest}.

Consider now case 1. We take a knot diagram $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ as flat knot limit of a knot K and another diagram $\mathcal{K}' = (\pi, \epsilon', \mathcal{I})$ as limit of K' with the sole difference that $\epsilon(c) = -\epsilon'(c)$ at a crossing $c \in \mathcal{C}$. Consider now the cylinder $U_c \times \mathbb{R}$. Within this cylinder the flat knot limits of the knots K and K' differ only by a parity transformation.

Hence the function h with arguments in $U_c \times \mathbb{R}$ is the same for K and K' and therefore independent of $\epsilon(c)$ and $\epsilon'(c)$ in the range $\Delta_3^{(1)}(c)$.

For case 2 with $x(s_2)$ and $x(s_3)$ encountering each other at c we consider equation (A.13). Since h is a continuous function in all three arguments the expression can be replaced by

$$(\lim_{\substack{t\to 0\\ t>0}} - \lim_{\substack{t\to 0\\ t<0}}) h(H_0 \circ x(s_1), H_t \circ x(s_2), H_t \circ x(s_3)) = 0. \tag{A.14}$$

The two limits correspond to flattening K to $\mathcal{K}$ and K' to $\mathcal{K}'$. Since the last expression vanishes, the function h is independent of $\epsilon(c)$ in the range $\Delta_3^{(2)}(c)$.

We now consider case 3. In the range Δ_3^{Rest} the function $h(H_t \circ x(s_1), H_t \circ x(s_2), H_t \circ x(s_3))$ is well defined for $t = 0$ because no two variables coincide at any crossing. The independence of h on the values of the crossing function is clear, because at every variable both limits $t \to 0$ lead to the same result, and because of the continuity of h one can set $t = 0$.

References

[1] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Comm. Math. Phys. **121** (1989) 351

[2] E. Guadagnini, *The Link Invariants of the Chern-Simons Field Theory*, de Gruyter, Berlin (1993)

[3] M. Alvarez and J.M.F. Labastida, *Numerical knot invariants of finite type from Chern-Simons perturbation theory*, Nucl. Phys. **B433** (1995) 555

[4] E. Guadagnini, M. Martellini, M. Mintchev, *Wilson lines in Chern-Simons theory and link invariants*, Nucl. Phys. **B330** (1990) 575

[5] W. B. R. Lickorish and Kenneth C. Millet, *A polynomial invariant of oriented links*, Topology **26** (1987) 107

[6] V.A. Vassiliev, *Cohomology of knot spaces*, in: *Theory of singularities and its applications*, (Adv. Sov. Math., vol. 1) Am. Math. Soc. (1990)

[7] Joan S. Birman and Xiao-Song Lin, *Knot polynomials and Vassiliev's invariants*, Invent. math. **111** (1993) 225;

[8] Joan S. Birman, *New points of view in knot theory*, Bull. Am. Math. Soc. **28** (1993) 253

[9] D. Rolfsen, *Knots and Links*, Publish or Perish (1976)